Project Description

1. Subject

The mathematics at the intersection of PDEs, Harmonic Analysis, and Geometric Measure Theory of the XXth century pushed the limits of regularity to discover optimal conditions under which the solutions of PDEs (and the corresponding physical phenomena) exhibited desired estimates. Some key geometric notions (e.g., rectifiability) and conjectures (e.g., Painlevé problem) go back more than 100 years. However, it was not until the beginning of the XXIst century, and even more so the last decade, that the situation was ripe for a dramatic breakthrough, and suddenly it became possible to obtain sharp results, which establish *equivalence* of scale-invariant geometric, analytic, and PDE properties of sets.

A good illustrative example is the celebrated converse to the 1916 theorem of F. and M. Riesz. The latter says that the harmonic measure is absolutely continuous with respect to the Hausdorff measure for a simple rectifiable curve in a plane. The converse, specifically, the fact that the absolute continuity of the harmonic measure implies rectifiability, was proved by J. Azzam, S. Hofmann, J.-M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa, and A. Volberg [**?**] in 2016, exactly 100 years after the F. and M. Riesz result. It furthermore opened a door to the final, necessary and sufficient conditions for the weak-A^∞ property of the harmonic measure obtained just a few months ago [**?**], [**?**].

These results did not appear in a vacuum and became a pinnacle of a long line of developments. It includes beautiful characterizations of uniform rectifiability in terms of square functions and boundedness of singular integrals by G. David and S. Semmes in the beginning of the 90s [**?**], the characterization in terms of the Cauchy transform by Mattila, Melnikov, and Verdera [**?**], the solution of the aforementioned Painlevé problem, and ultimately, the solution of the David-Semmes conjecture about the equivalence of uniform rectifiability and boundedness of the Riesz transforms by Nazarov, Tolsa, and Volberg, in 2015. A less PDE inclined reader could call each of these results a pinnacle by itself, and we will be discussing the main ideas behind some of them in the proposed lecture series.

Our main goal, however, is more ambitious than just that. Ever since this string of spectacular developments had largely settled the situation for domains with $n-1$ dimensional boundaries, it has been clear that virtually none of the proofs can be extended to lower dimensional sets, in part due to the notorious lack of a meaningful PDE theory comparable in power and scope to the classical case. Indeed, harmonic functions do not “see" lower-dimensional boundaries, for instance, a curve in $\mathbb{R}^3$ would be completely ignored, which undermines both PDE and analytic approaches, as the special role of the Cauchy transform or, in higher dimensions, the Riesz transform, in the theory relies on their harmonicity in essential ways.

In 2015 the main lecturer, Svitlana Mayboroda, together with Guy David (Université Paris XI), has launched the theory devoted to the analysis, PDEs, and associated geometric measure theory in domains with lower dimensional boundaries [**?**, **?**, **?**]. In the intervening years the project has grown into a large collaboration, involving, in particular, J. Feneuil (Temple), M. Engelstein (MIT), Z. Zhao (IAS/Chicago), Z. Dai and B. Poggi (UMN), to mention only some key contributors, and has turned out to be a compelling and fascinating world of its own, connected in unprecedented ways to geometric analysis, nonlinear PDEs, and nonlocal singular operators. Considerably beyond a "natural" extension of classical results in co-dimension one (mostly linear and local and their nature), the authors have discovered some dramatically new phenomena and challenges; for instance the “magic distance" function - an explicit Green function for a certain key operator in an arbitrary domain, new geometric characterizations of rectifiability via derivatives of this distance, connections with the Caffarelly-Silvestre extension operator and fractional Laplacian, analogies with p-Laplacian and, on the other end, with the behavior of Paneitz operators and other objects in geometric/microlocal analysis, and many others. While heavily inspired by the aforementioned success of co-dimension 1 results, this theory is, in the end, quite different, exhibiting new challenges but also new rich interactions with other areas of PDEs and analysis. The theory is growing exponentially and we hope that the proposed lectures will provide a structured encounter of the substantial body of new results, highlight the key achievements and great open problems, and attract young researchers to this quickly developing field.

Finally, it is important to mention that these problems have surprising and intricate applications across several areas of physics, materials science, and engineering. The principal lecturer is heading two big interdisciplinary projects, Simons Collaboration on Wave Localization, and NSF-funded RAISE project on the hidden structure of the disorder in quantum systems. The material featured in the proposed lecture series includes portions of the research pertaining to this interdisciplinary work, and the audience will have a unique chance to have a direct exposure to the ways in which seemingly abstract concepts and results at the cutting edge of pure mathematics can immediately influence state-of-the-art engineering of photonic devices and the physics behind them.

Let us discuss a brief structure of the lecture series.

2. Description of lectures

Lecture 1. We will start focusing on the classical setting of the domains with $n-1$ dimensional boundaries and discuss the recent breakthroughs pertaining to equivalence of scale-invariant geometric, analytic, and PDE properties of sets, from the PDE perspective. This includes introduction of the key concepts: harmonic functions, harmonic measure (and its importance in PDEs, probability, analysis), rectifiability, all the way to the state-of the-art results and open problems concerning the dimension and structure of the harmonic measure. At this point, we concentrate more on the intuition and statements of the key results and problems, rather than the in-depth proofs.

Lecture 2. Much as in Lecture 1, we provide a high-level overview of the results for $n-1$ dimensional surfaces, now from the Analysis perspective. We discuss the celebrated David-Semmes conjecture (now the theorem due to Nazarov, Tolsa, and Volberg) regarding equivalence of L^2 boundedness of Riesz transform and uniform rectifiability, its prototype for the Cauchy transform by Mattila, Melnikov, and Verdera, connections with the Painlevé problem and Vitushkin conjecture, and the intuition behind this circle of results. We will give the gist of the proof of equivalence between boundedness of the Cauchy transform and rectifiability, featuring the Menger curvature, as it is perhaps the simplest and most illustrative example of clear equivalence between a geometric (curvature) and analytic (L^2 norm of a singular integral) characteristization of a set.

Lecture 3. At this point we are ready to discuss the challenges of extending any of the aforementioned results to sets of the dimension lower than $n-1$ in $\mathbb{R}^n$, for instance, curves in $\mathbb{R}^3$. We explain why the harmonic functions are not suitable and what fails in the context of the Riesz transform. We introduce our new main heroes: the degenerate elliptic operators, explain the rules of the homogeneity and other aspects of intuition in the lower-dimensional world, and extend the scope further to feature the generalization of the Caffarelli-Silvestre extension operator, discuss its connections with the fractional Laplacian, highlight what are the key known properties and challenges for this type of degenerate operators (even in co-dimension one). To build up the intuition, we derive explicit formulas for solutions of our equations for some simple domains, such as $\mathbb{R}^n \setminus \mathbb{R}^d$.

Lecture 4. This lecture is devoted to the basic elliptic theory. Considering the full generality of degenerate elliptic operators and Ahlfors-David regular sets, possibly of mixed dimension, we develop the theory of Sobolev spaces, trace and extension theory, prove existence and uniqueness of solutions, maximum principle, Harnack inequality, Moser estimates, establish existence of the harmonic measure, key estimates for the Green function, and comparison principle. We assume a basic familiarity of the audience with these notions for harmonic functions (at the level of a standard PDE course), and highlight the differences with the lower-dimensional setting, particularly surprising at a certain topological level. At this point, most news are good news: one can establish the elliptic theory considerably beyond the limits of what was previously deemed possible, provided the homogeneity of weights is carefully (but naturally) arranged.

Lecture 5. We pass on to the question of absolute continuity of the resulting “harmonic" (elliptic) measure with respect to the Hausdorff measure in lower dimensional sets. Now, however, we face a series of bad surprises (counterexamples to certain natural conjectures involving the Euclidean distance) and this leads to the explanation for a somewhat unconventional choice of the "Laplacian" – the new key elliptic operator in our theory, relying, in turn, on a new notion of distance. We discuss the key points of the proof of the absolute continuity of the elliptic measure with respect to the Hausdorff measure on a Lipschitz graph with a small Lipschitz constant highlighting, once again, the new aspects of it: a non-trivial role of torsion around the boundary (non-existent in co-dimension one), the new change of variables, and other key ideas in the argument.

Lecture 6. Having established the result for the small Lipschitz graphs, we now attack absolute continuity of the elliptic measure with respect to the Hausdorff measure on any d dimensional uniformly rectifiable boundary. This lecture will familiarize the audience with “big pieces" techniques, shielding by means of construction of saw-tooth domains, controlled replacements of bad pieces, extrapolation, and other elements of the proof, each of them being prominent and important on their own right. We will also discuss how (quite remarkably) this result is, in some sense, stronger than the analogous one in co-dimension one, for the latter needs additional topological assumptions due to the Bishop-Jones counterexample.

Lecture 7. We come back to the discussion of the newly introduced distance function and present the characterizations of rectifiability in terms of the Carleson measure property of derivatives of this distance (new and important even in co-dimension one). We further present the emerging analogue of the Riesz tranform characterization for lower-dimensional sets, nonlocal and non-linear, hypersingular in a certain sense, yet miraculously resonating (even superficially coinciding) with the David-Semmes conjecture from Lecture 2 in co-dimension one.

Lecture 8. We present one of the most mysterious new features of sets with lower-dimensional boundaries: for a certain operator, a natural analogue of the Laplacian, our distance function is the Green function with the pole at infinity, for an arbitrary domain with a d-dimensional boundary, $d < n-2$. This is a very surprising discovery (the PDE theory has only a few examples of explicit Green functions, and all of them come from very simple domains) and a very powerful one, as we will see shortly. For now, we discuss why this discovery rends the converse to Lecture 6, a property that absolute continuity of harmonic measure implies rectifiability, impossible at least for one operator, again in dramatic contrast with co-dimension one.

Lecture 9. In the search of the "if and only if" conditions, some of which can fail in sharp contrast with the classical theory as the previous lecture shows, we turn to a new feature of our setting: rotation invariance of solutions. It is far from being clear even how to define rotation invariance around a rectifiable curve, and yet it is quite natural to expect that the solution is virtually rotation invariant in a set close to $\mathbb{R}^n \setminus \mathbb{R}^d$. It turns out that a certain correct notion of rotation invariance for the Green function can, in fact, characterize rectifiability, and could even be extended to give new estimates for the classical, harmonic, Green function, even though no rotation per se is relevant.

Lecture 10. We briefly list other currently available results and then discuss a long array of fascinating open problems, from well-known named conjectures to some potential game-changers that surfaced in the light of recent discoveries.

3. A READING LIST

Elliptic theory, including domains with lower dimensional boundaries:

- Guy David, Joseph Feneuil, Svitlana Mayboroda, *Elliptic theory in domains with boundaries of mixed dimension*, submitted, https://arxiv.org/abs/2003.09037
- Guy David, Joseph Feneuil, Svitlana Mayboroda, *Elliptic theory for sets with higher co-dimensional boundaries.* Mem. Amer. Math. Soc. 274 (2021), no. 1346, vi+123 pp.
- Guy David, Joseph Feneuil, Svitlana Mayboroda, *A new elliptic measure on lower dimensional sets*, Acta Math. Sin. (Engl. Ser.), the special issue in honor of the 65th birthday of Professor Carlos Kenig, 35 (2019), no. 6, 876–902.
- Carlos Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems.* CBMS Regional Conference Series in Mathematics, 83. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994.

Ahlfors regularity, rectifiability, uniform rectifiability:

- Guy David, Stephen Semmes, *Analysis of and on uniformly rectifiable sets.* Mathematical Surveys and Monographs, 38. American Mathematical Society, Providence, RI, 1993.
- Pertti Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995.

Distance function:

- Guy David, Max Engelstein, Svitlana Mayboroda, *Square functions, non-tangential limits and harmonic measure in co-dimensions larger than one,* Duke Math. J.

170 (3) 455–501, 15 February 2021.
https://doi.org/10.1215/00127094-2020-0048

Absolute continuity of harmonic measure, well-posedness of boundary problems, and related matters. Domains with co-dimension one boundaries:

- Jonas Azzam, Steve Hofmann, José María Martell, Svitlana Mayboroda, Mihalis Mourgoglou, Xavier Tolsa, and Alexander Volberg, *Rectifiability of harmonic measure*, Geom. Funct. Anal. 26 (2016), no. 3, 703–728. DOI: 10.1007/s00039-016-0371-x
- Steve Hofmann, Jose Maria Martell, Svitlana Mayboroda, *Uniform Rectifiability, Carleson measure estimates, and approximation of harmonic functions*, Duke Math. J. 165 (2016), no. 12, 2331–2389.
- Steve Hofmann, José María Martell, Svitlana Mayboroda, *Transference of scale-invariant estimates from Lipschitz to Non-tangentially accessible to Uniformly rectifiable domains*, submitted,
https://arxiv.org/abs/1904.13116
- Jonas Azzam, Steve Hofmann, José María Martell, Mihalis Mourgoglou, Xavier Tolsa, *Harmonic measure and quantitative connectivity: geometric characterization of the Lp-solvability of the Dirichlet problem,* Invent. Math. 222 (2020), no. 3, 881–993.
- Guy David, Svitlana Mayboroda, *Good elliptic operators on Cantor sets*, Advances in Mathematics, Volume 383, 2021, 107687,
https://doi.org/10.1016/j.aim.2021.107687
- Steve Hofmann, José María Martell, Svitlana Mayboroda, Tatiana Toro, Zihui Zhao, *Uniform rectifiability and elliptic operators satisfying a Carleson measure condition*, Geom. Funct. Anal. 31, 325–401 (2021).
https://doi.org/10.1007/s00039-021-00566-4, https://arxiv.org/pdf/2008.04834.pdf
- Steve Hofmann, José María Martell, *Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in Lp,* Ann. Sci. Éc. Norm. Supér. (4) 47 (2014), no. 3, 577–654.
- Steve Hofmann, José María Martell, Ignacio Uriarte-Tuero, *Uniform rectifiability and harmonic measure, II: Poisson kernels in Lp imply uniform rectifiability,* Duke Math. J. 163 (2014), no. 8, 1601–1654.

Absolute continuity of harmonic measure, well-posedness of boundary problems, and related matters. Domains with lower-dimensional boundaries:

- Guy David, Joseph Feneuil, Svitlana Mayboroda, *Dahlberg's theorem in higher co-dimension*, J. Funct. Anal. 276 (2019), no. 9, 2731–2820.
- Guy David, Svitlana Mayboroda, *Harmonic measure is absolutely continuous with respect to the Hausdorff measure on all low-dimensional uniformly rectifiable sets*, IMRN, accepted, https://arxiv.org/abs/2006.14661
- Joseph Feneuil, *Absolute continuity of the harmonic measure on low dimensional rectifiable sets,* https://arxiv.org/pdf/2006.03118.pdf
- Svitlana Mayboroda, Bruno Poggi, *Carleson perturbations of elliptic operators on domains with low dimensional boundaries*, Journal of Functional Analysis, Volume 280, Issue 8, 2021, 108930,
https://doi.org/10.1016/j.jfa.2021.108930.

- Svitlana Mayboroda, Zihui Zhao, *Square function estimates, BMO Dirichlet problem, and absolute continuity of harmonic measure on lower-dimensional sets*, Analysis & PDE, 12 (2019), no. 7, 1843–1890.
- Joseph Feneuil, Svitlana Mayboroda, Zihui Zhao, *Dirichlet problem in domains with lower dimensional boundaries*, Revista Matemática Iberoamericana, Volume 37, Issue 3, 2021, pp. 821–910.
  DOI: 10.4171/rmi/1208, https://arxiv.org/pdf/1810.06805.pdf

Green function characterizations of (uniform) rectifiability:

- Guy David, Svitlana Mayboroda, *Approximation of Green functions and domains with uniformly rectifiable boundaries of all dimensions*, submitted, https://arxiv.org/pdf/2010.09793.pdf
- Guy David, Linhan Li, Svitlana Mayboroda, *Carleson measure estimates for the Green function*, Archive for Rational Mechanics and Analysis, accepted, https://arxiv.org/pdf/2102.09592.pdf
- Guy David, Linhan Li, Svitlana Mayboroda, *Carleson estimates for the Green function on domains with lower dimensional boundaries*, submitted, http://arxiv.org/abs/2107.08101
- Guy David, Joseph Feneuil, Svitlana Mayboroda, *Green function estimates on complements of low-dimensional uniformly rectifiable sets*, Math. Ann., accepted, https://arxiv.org/pdf/2101.11646.pdf

4. Basic definitions relevant to the upcoming conference

The lecturer will make an effort to make the presentation self-contained as much as possible. It is expected that the audience has a basic command of Real Analysis, PDEs, and Harmonic Analysis at least at the level of the coverage of a graduate-level course. It is advised that the audience is comfortable with the definitions and results presented in the introduction (Part I: Background information and the statements of the main results) to the book

- Guy David, Stephen Semmes, *Analysis of and on uniformly rectifiable sets.* Mathematical Surveys and Monographs, 38. American Mathematical Society, Providence, RI, 1993.

as well as the material reviewed in

- Guy David, Joseph Feneuil, Svitlana Mayboroda, *A new elliptic measure on lower dimensional sets*, Acta Math. Sin. (Engl. Ser.), the special issue in honor of the 65th birthday of Professor Carlos Kenig, 35 (2019), no. 6, 876–902,

at least in the setting of the domains with co-dimension one boundaries. The aforementioned reference is a condensed version containing definitions and results only, and the reader can consult references therein for the full proofs. The key expanded reference for this background material is

- Guy David, Joseph Feneuil, Svitlana Mayboroda, *Elliptic theory in domains with boundaries of mixed dimension*, submitted, https://arxiv.org/abs/2003.09037

Finally, a beautiful exposition of the historical context of the related matters is given in

- Carlos Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems.* CBMS Regional Conference Series in Mathematics, 83. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994.